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On Approximately Optimal H^∞ Controllers for Distributed Systems*

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Abstract

In this paper we elucidate the structure of all the approximately optimal H^∞ controllers for stable distributed plants with rational weights. We identify the finite and infinite dimensional parts of these H^∞ controllers. It is shown that one can obtain a finite dimensional approximately optimal controller by appropriately approximating the infinite dimensional part of the optimal controller. Also, it is possible to find certain bounds on the deviation from the optimal performance using this procedure.

1 Introduction

In this paper we consider the one block H^∞ sensitivity minimization problem for SISO infinite dimensional systems. Our main purpose is to develop a method for obtaining finite dimensional approximately optimal H^∞ controllers. We use the results of [5] to describe the structure of all approximately (sub)-optimal H^∞ controllers. In the case of rational weights and distributed stable plants we will be able to identify the finite and infinite dimensional parts of the controller. A natural way of obtaining a finite dimensional controller is to approximate the infinite dimensional part of the optimal controller.

It is known that H^∞-optimal controllers for finite dimensional plants with rational weights are finite dimensional. Therefore, another way to obtain a finite dimensional H^∞ controller is to approximate the original plant with a finite dimensional system, compute the optimal controller for this approximate system, and then check whether this controller yields a performance close to the optimum for the original plant. However, it is obvious that there is no guarantee that the optimal controller of the approximate system will even stabilize the original plant. See [2] and [9] for all the details about this method and the difficulties associated with it.

The techniques and results of this paper are valid for a large class of stable distributed plants. However, when we demonstrate our method in detail with an example, we will specialize to delay systems. For such systems the approach of ([4]) is similar to the one given here. We have applied the methods given below to a flexible beam problem in some joint work with Kathryn Lenz Peckham and Blaise Morton [7].

The rest of the paper is organized as follows. In the next section we summarize the main results of [5], where the main idea is to use the one step extension theory of [1] to characterize the suboptimal solutions to the generalized interpolation problem. In Section 3 we exploit this characterization to illustrate the structure of the optimal and approximately optimal H^∞ controllers. We apply our procedure to obtain finite dimensional controllers, for distributed stable plants with invertible outer parts in Section 4; strictly proper plants are considered in Section 5; and a design example is given in Section 6 to illustrate how to deal with this situation. Finally, in the last section we summarize the results of the paper and make some concluding remarks.

Notation: Our notation is standard. All Hardy spaces of this paper are defined on the unit disc in the usual way. In particular, we consider the systems as transfer functions (of the complex variable z i.e. the Z transform variable). For continuous time systems we can think of this as transfer functions (of the Laplace transform variable s), transformed via bilinear transformation $s = \frac{1}{\tau}\frac{1+z}{1-z}$, $\tau > 0$, that maps the unit disc to the right half plane.

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2 Preliminary Remarks

The sensitivity minimization problem is to find an internally stabilizing controller C such that the following optimum performance is achieved

$$\inf_{C \text{ stabilizing}} \|W(1+PC)^{-1}\|_\infty =: \mu.$$

See Figure 1 for the closed loop set-up, where P is the plant to be controlled and W is the weight modelling the disturbances. Assuming that the weight is an outer function and the plant P is stable we can transform this problem to a Nehari problem (in the usual way of first invoking the Youla parametrization for the controller $C = Q_c(1-PQ_c)^{-1}$, $Q_c \in H^\infty$, $(1-PQ_c) \neq 0$; and then finding an inner/outer factorization for $P = mP_o$, where m is inner and P_o is outer):

$$\mu = \inf_{Q \in H^\infty} \|W - mQ\|_\infty. \tag{1}$$

Conversely, from (1) by finding Q realizing μ, and by inverting W and P_o, we get the optimal controller C_0 which internally stabilizes the system and satisfies

$$\|W(1+PC_0)^{-1}\|_\infty = \mu. \tag{2}$$

Given a tolerance $\epsilon > 0$, we say that C_ϵ is ***approximately optimal*** (or ***suboptimal***), with tolerance ϵ, if it internally stabilizes the system and satisfies the bound

$$\|W(1+PC_\epsilon)^{-1}\|_\infty \leq \mu + \epsilon =: \rho. \tag{3}$$

One important point which should be emphasized is that when the plant is strictly proper its outer part P_o is only approximately invertible in H^∞ as a stable causal transfer function, so a proper optimal controller does not in general exist. Nevertheless, even if P_o is infinite dimensional, there are good rational approximations for the inverse of such outer functions, that can be used in the implementation of an approximately optimal controller. However, this is not the only problem in computing the H^∞ controllers. The issue to be discussed first in this paper is what happens when the inner part m of the plant is infinite dimensional. So, in Section 4 the question of approximating the inverse of P_o will be left aside, and it will be assumed that P_o^{-1} is in H^∞. Then, in Sections 5 and 6 we consider the case where P_o is only approximately invertible.

The difficulty in the case of infinite dimensional m comes from the fact that C_0 in (2) is infinite dimensional. Hence, implementation of the optimal controller is not easy. Another (possibly more serious) problem is that C_0 is very sensitive to the parameters of m. In other words, if we use approximate values for those parameters (instead of the exact ones) in the controller, the resulting closed loop system may not be stable.

In the light of the above discussion we now assume that m is infinite dimensional, and consider the following problem: given $\rho \geq \mu$, find the set of all $Q \in H^\infty$ such that

$$\|W - mQ\|_\infty \leq \rho. \tag{4}$$

Let us summarize the results of [5] in connection with the above problem. Suppose that the weight is rational: $W(z) = p(z)/q(z)$ where $p(z) = p_0 + zp_1 + \ldots + z^n p_n$ and $q(z) = q_0 + zq_1 + \ldots + z^n q_n$, (i.e. n is the maximum of the degrees of p and q, so some of the above coefficients may well be zero). Let S denote the unilateral shift on H^2 and define the space $H(m) = H^2 \ominus mH^2$. Then *the compressed shift* associated with $H(m)$ is defined as $T := P_{H(m)}S|_{H(m)}$, where $P_{H(m)}$ denotes orthogonal projection.

First, consider the optimal case: $\rho = \mu$. The optimal interpolant Q_{opt}, which makes $\|B_{opt}\|_\infty = \rho$, where

$$B_{opt} = W - mQ_{opt},$$

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can be computed using Sarason's theorem ([8]) which states that

$$\mu = \|W(T)\|, \quad W(T) := p(T)q(T)^{-1}.$$

The *essential norm* can be defined as follows:

$$\|W(T)\|_e = \sup\{|W(\zeta)| \ : \ \zeta \text{ singular point of } m\}.$$

We need to assume $\mu > \|W(T)\|_e$, see [6], to conclude that $W(T)$ attains its norm at a singular value $\rho = \mu$. In this case there exists a singular vector h_o, for the so-called ([3]) *skew Toeplitz* operator

$$A_\rho := \rho^2 q(T)q(T)^* - p(T)p(T)^*$$

(* denotes adjoint) which makes

$$A_\rho h_o = 0.$$

The vector h_o can be computed explicitly from the problem data $W = p/q$ and m in terms of a determinantal formula; see [5], and [6]. Then, B_{opt} can be found via Sarason's result as

$$B_{opt} = \rho^2 \frac{q(T)^* h_o}{p(T)^* h_o}.$$

Let us now consider the case where: $\rho > \mu$. It is obvious that in this case A_ρ is invertible and its inverse can be computed explicitly; again, the formula is given in [5]. This is going to be used in the characterization of all the suboptimal solutions $Q_s \in H^\infty$ which make

$$\|W - mQ_s\|_\infty \le \rho. \tag{4}$$

This characterization is obtained using the one step extension procedure of [1]. Here we want to summarize the method briefly. Set $m_v(z) := zm(z)$ and let T_v denote the compression of S to $H(m_v) = H(m) \oplus \mathbf{C}z$. For $\alpha \in \mathbf{C}$ fixed, the problem of finding $B_{opt}(z, \alpha) = (W - \alpha m - m_v Q_{opt}^\alpha)(z)$ such that

$$\|B_{opt}(\cdot, \alpha)\|_\infty = \|(W - \alpha m)(T_v)\| = \rho$$

can be solved using the technique described above for the optimal case. From the one step extension theory ([1]) we know that the set of all such $\alpha \in \mathbf{C}$ form a circle, say Γ. Furthermore, the equation of Γ can be explicitly calculated. Then, the set of all suboptimal solutions $Q_s \in H^\infty$ satisfying (4) is obtained in terms of $B_{opt}(z, \phi(u))$:

$$W - mQ_s = B_{opt}(z, \phi(u)),$$

where $\phi(z)$ is a linear fractional map taking the unit circle to Γ, and $u \in H^\infty$, $\|u\|_\infty \le 1$ is the free parameter. The explicit characterization is as follows. Set

$$g_1 := (\rho^2 q(T) P_{H(m)} q(S)^* - p(T) P_{H(m)} p(S)^*) m,$$

$$g_2 := q_0 p(T)(1 - m\, m(0)),$$

and

$$h_1 := A_\rho^{-1} g_1, \quad h_2 := A_\rho^{-1} g_2.$$

For a given $\alpha \in \Gamma$ define

$$h_\alpha(z) := m(z) - h_1(z) - \overline{\alpha} h_2(z),$$

and

$$B(z, \alpha) := \frac{\rho^2 q(S)^* h_\alpha}{p(S)^* h_\alpha - \overline{\alpha} q_0}.$$

Then we have the following result.

Theorem 1([5]) The set of all functions of the form

$$B(z) = W(z) - m(z)Q_s(z)$$

with $Q_s \in H^\infty$, such that $\|B\|_\infty \le \rho$, is given by

$$\{B(z, \alpha) = \frac{\rho^2 q(S)^* h_\alpha}{p(S)^* h_\alpha - \overline{\alpha} q_0} |_{\overline{\alpha} = \eta + r/u}, u \in H^\infty \|u\|_\infty \le 1\}$$

where r and η are certain explicitly computable constants. See [5] for the formulae.

3 Structure of the suboptimal controllers

From the above parametrization we are going to obtain the structure of all suboptimal H^∞ controllers. Using the notation of Theorem 1, we set $B_\alpha(z) := B(z, \alpha)$. We can find the controller from $C = Q_c(1 - PQ_c)^{-1}$, the Youla parametrization, where Q_c is such that

$$B_\alpha = W - PWQ_c.$$

Therefore,

$$C = P^{-1}(B_\alpha^{-1} W - 1).$$

We now study B_α,

$$B_\alpha(z) = \frac{\rho^2 \tilde{q}(z) h_\alpha(z) - \rho^2 h_q(z)}{\tilde{p}(z) h_\alpha(z) - h_p(z) - q_0 z^n \overline{\alpha}}$$

where $h_q(z)$ and $h_p(z)$ are polynomials of degree $\le n - 1$ and $\tilde{q}(z) = z^n q(z^{-1})$, similarly $\tilde{p}(z) = z^n p(z^{-1})$. Then,

$$P_o C = \frac{1}{m}\left(\frac{\tilde{p}(z) h_\alpha(z) - h_p(z) - q_0 z^n \overline{\alpha}}{\rho^2 \tilde{q}(z) h_\alpha(z) - \rho^2 h_q(z)} \ \frac{p(z)}{q(z)} - 1\right)$$

$$= \frac{1}{m}\left(\frac{-\lambda(z) h_\alpha(z) - p(z) h_p(z) + \rho^2 q(z) h_q(z) - q_0 z^n p(z)\overline{\alpha}}{(\rho^2 \tilde{q}(z) h_\alpha(z) - \rho^2 h_q(z)) q(z)}\right),$$

where $\lambda(z) = \rho^2 \tilde{q}(z) q(z) - \tilde{p}(z) p(z)$. Recall that $h_\alpha(z) = m(z) - h_1(z) - \overline{\alpha} h_2(z)$. It is easy to see from the inversion of the skew Toeplitz operator A_ρ, that h_1 and h_2 have the following form (see e.g. Lemma 2.1 and Corollary 2.5 of [5])

$$h_1(z) = \frac{f_1(z) + m(z) F_1(z)}{\lambda(z)},$$

and

$$h_2(z) = \frac{f_2(z) + m(z) F_2(z)}{\lambda(z)}$$

for some f_1, F_1, f_2, F_2 polynomials of degree $\le 2n$. This leads us to the following expression:

$$P_o C = \left(\frac{-\lambda(z)}{\rho^2 q(z) \tilde{q}(z)}\right) \frac{G_u(z)}{1 + m(z) G_u(z)} \tag{5a}$$

where

$$G_u(z) = \tilde{q}(z) \frac{F_\alpha(z) - \lambda(z)}{\tilde{q}(z) f_\alpha(z) + h_q(z) \lambda(z)}$$

$$F_\alpha(z) := F_1(z) + \overline{\alpha} F_2(z), \text{ and } f_\alpha(z) := f_1(z) + \overline{\alpha} f_2(z).$$

and

$$\overline{\alpha} = \frac{r}{u} + \eta.$$

Note that

$$\frac{-\lambda(z)}{\rho^2 q(z) \tilde{q}(z)} = \frac{p(z) \tilde{p}(z)}{\rho^2 q(z) \tilde{q}(z)} - 1 = \frac{W(z) W(z^{-1})}{\rho^2} - 1.$$

We summarize the above formulae with the following.

Corollary 1 The set of all controllers which internally stabilize the plant P, and satisfy the bound

$$\|W(1 + PC)^{-1}\|_\infty \le \rho$$

for $\rho \ge \mu$, have the form

$$C = \left(\frac{W(z) W(z^{-1})}{\rho^2} - 1\right) \frac{G_u(z)}{1 + m(z) G_u(z)} P_o^{-1}(z)$$

$u \in H^\infty$, $\|u\| \le 1$, where $G_u(z)$ is a linear fractional transformation in the free parameter u:

$$G_u(z) = \frac{\varphi_1(z) + \varphi_2(z) u}{\varphi_3(z) + \varphi_4(z) u}$$

with $\varphi_1, \ldots, \varphi_4$ polynomials of degree $\le 3n$. They can be computed explicitly from the equations given in [5], via f_1, F_1, f_2, F_2, r and η. □

Figure 2 shows the block diagram of the closed loop system with an approximately optimal controller. From the structure of the controller we see that if the free parameter u is chosen as a finite dimensional transfer function then the only infinite dimensional part of the controller is m and it appears at the feedback path around G_u. We thus identify the finite and infinite dimensional parts of the controller. Note that in order to have a finite dimensional controller we must choose the free parameter u depending on m, moreover u itself must be infinite dimensional so that the infinite dimensional parts of the controller gets cancelled.

As it can be seen from the above formula for C, it is not easy to characterize the set of all $u \in H^\infty$ and $\|u\| \leq 1$, such that the transfer function

$$\frac{G_u}{1+mG_u}$$

is rational. Therefore, we are going to follow a different approach to obtain finite dimensional approximately optimal H^∞ controllers. This is the subject of the next section.

4 Finite dimensional approximately optimal controllers

Recall from the Theorem 1 that the structure described in the Corollary 1 is valid for the optimal controller as well. In the case of the optimal controller, however, the free parameter is absent in the term G_u; that is, instead of G_u we have a fixed rational function, say G. Hence, if we replace m by a rational function m_f in the feedback path around G we obtain a finite dimensional controller. We now study the effects of this approximation of m by m_f. More specifically we want to answer the following questions: under which conditions the stability is preserved, and what is the deviation from the optimal performance?

For simplicity of the notation and the computations, we will restrict ourselves to the following case:

$$W(z) = \frac{p(z)}{q(z)}, \quad q(z) = 1, \quad p(z) = p_0 + p_1 z,$$

and $m(z)$ is any arbitrary inner function. Then, the optimal controller can be computed as (see Appendix A),

$$C(z) = \left(\frac{W(z)W(z^{-1})}{\rho^2} - 1\right) \frac{G(z)}{1+m(z)G(z)} P_o^{-1}(z) \qquad (5b)$$

with $G(z) = \frac{\rho z}{p(z)}$.

Consequently, the optimal sensitivity is

$$B_{opt}(z) = W(z) \frac{1+m(z)\rho z/p(z)}{1+m(z)\tilde{p}(z)/\rho}.$$

Now we replace m by m_f in the expression for the controller, so that the controller becomes a finite dimensional transfer function:

$$C_f(z) := \left(\frac{W(z)W(z^{-1})}{\rho^2} - 1\right) \frac{\rho z/p(z)}{1+m_f(z)\rho z/p(z)} P_o^{-1}(z).$$

It is easy to see that if we use C_f, in the closed loop as a controller, then the sensitivity function becomes

$$B_f(z) = W(z) \left(\frac{1+m(z)\rho z/p(z)+\Delta}{1+m(z)\tilde{p}(z)/\rho+\Delta}\right).$$

where $\Delta = (m_f(z) - m(z))\rho z/p(z)$.

Set

$$R(z) := \frac{p(z)}{\rho z}(1+m(z)\frac{\tilde{p}(z)}{\rho}) \text{ and } \Delta_m(z) := m_f(z) - m(z).$$

Then we can rewrite B_f as

$$B_f(z) = B_{opt}(z)\frac{R(z)}{R(z)+\Delta_m(z)} + W(z)\frac{\Delta_m(z)}{R(z)+\Delta_m(z)}.$$

This expression shows that a rational function m_f, which makes C_f approximately optimal, can be found by studying the relation between the terms R and Δ_m.

From this point on, in the examples that we are going to consider, we will conduct our analysis and design in the right half plane, which is more natural for continuous time systems. When we do this we transform the problem data by using the conformal map $z = \frac{\tau s-1}{\tau s+1}$ between the right half plane and the unit circle. In particular $\hat{R}(s)$ denotes $R(z)|_{z=\frac{\tau s-1}{\tau s+1}}$, and $\Delta_{\hat{m}}(s) := \Delta_m(z)|_{z=\frac{\tau s-1}{\tau s+1}}$, and similarly for all the other transfer functions. Let us now compare $\hat{R}$ and $\Delta_{\hat{m}}$ to analyze the approximate optimality of $\hat{C}_f$.

First of all in order to guarantee stability we should have

$$\hat{R}(s) + \Delta_{\hat{m}}(s) \neq 0$$

inside the closed right half plane. Also, since we are looking for a performance close to the optimum, the H^∞ norm of $\hat{B}_f$ should be close to ρ. Note that if we could make $|\hat{R}(j\omega)| \gg |\Delta_{\hat{m}}(j\omega)|$ for all $\omega \geq 0$, then we would have $|\hat{B}_f(j\omega)| \approx |\hat{B}_{opt}(j\omega)| \quad \forall \omega \geq 0$ which impies that $\|\hat{B}_f\|_\infty \approx \|\hat{B}_{opt}\|_\infty = \rho$. However this is not possible in general, because there is no good uniform (on the imaginary axis) rational approximation for an irrational inner function which has essential singularities on the boundary. This is the main difficulty in finding the finite dimensional approximately optimal H^∞ controllers for distributed systems with invertible outer part. In the next section, we generalize the above idea, of designing $\hat{m}_f$ by comparing $\hat{R}$ with $\Delta_{\hat{m}}$, to plants which can be approximated uniformly on the imaginary axis.

5 On the outer part of the plant

In this section we consider the case where the plant $P(z)$ is continuous on the unit circle. In other words the outer part of the plant is such that the essential singularities of the inner part gets killed. For example if a plant with transportation delay has strictly proper outer part P_o, then it becomes continuous on the unit circle. This kind of plants can be approximated perfectly (up to a certain tolerance) by finite dimensional transfer functions. However, in this situation P_o is not invertible in H^∞, so we must find an approximate inverse.

Recall the structure of the optimal controller:

$$C(z) = \left(\frac{W(z)W(z^{-1})}{\rho^2} - 1\right) \frac{G(z)}{1+m(z)G(z)} P_o^{-1}(z).$$

We can rewrite this as

$$C(z) = \left(\frac{W(z)W(z^{-1})}{\rho^2} - 1\right) \frac{G(z)P_o^{-1}(z)}{1+P(z)G(z)P_o^{-1}(z)}.$$

A finite dimensional controller can be obtained by approximating P and P_o^{-1} separately:

$$C_f(z) = \left(\frac{W(z)W(z^{-1})}{\rho^2} - 1\right) \frac{G(z)P_{of}^{-1}(z)}{1+P_f(z)G(z)P_{of}^{-1}(z)},$$

where P_f and P_{of}^{-1} are finite dimensional proper approximations for P and P_o^{-1} respectively. See Figure 3 for the implementation of C_f.

Let us now analyze the performance of the closed loop system under this finite dimensional controller. After a simple algebra similar to the one in Section 4 we see that the sensitivity function $B_f := W(1+PC_f)^{-1}$ is in the form

$$B_f(z) = B_{opt}(z)\frac{R(z)}{R(z)+\Delta(z)} + W(z)\frac{\delta(z)}{R(z)+\Delta(z)}, \qquad (6)$$

where

$$R(z) := \left(1 + \frac{p(z)\tilde{p}(z)}{\rho^2 q(z)\tilde{q}(z)} m(z)G(z)\right) G(z)^{-1},$$

$$\delta(z) := m(z)\delta_{of}(z) + \delta_p(z)P_{of}^{-1}(z),$$

$$\Delta(z) := \frac{p(z)\tilde{p}(z)}{\rho^2 q(z)\tilde{q}(z)} m(z)\delta_{of}(z) + \delta_p(z)P_{of}^{-1}(z),$$

$$\delta_{of}(z) := P_o(z)P_{of}^{-1}(z) - 1, \quad \delta_p(z) := P_f(z) - P(z).$$

Following the ideas of Sections 4 and 5, to make C_f approximately optimal we can design P_f and P_{of}^{-1} by comparing R with Δ and δ. One important point to note is that since $P(e^{j\phi})$ is continuous on $\phi \in [-\pi, 0]$, we can approximate it up to a given tolerance by a finite dimensional transfer function P_f, uniformly on the unit circle. Also we can choose a proper rational P_{of}^{-1} such that $P_o P_{of}^{-1}$ is close to 1 on the unit circle excluding some arbitrarily small neighborhood of the point e^{j0}. Moreover, when P_o is strictly proper and P_{of}^{-1} is proper we have $\delta_{of}(e^{j\phi}) \to -1$ and $\delta_p(e^{j\phi})P_{of}^{-1}(e^{j\phi}) \to 0$ as $\phi \to 0$. Then, it is not difficult to see that $|B_f(e^{j\phi}) - W(e^{j\phi})| \to 0$ as $\phi \to 0$. On the other hand, when P_o is strictly proper, by definition of μ in (1), we necessarily have $|W(e^{j0})| \leq \mu$. Therefore, in this case having a proper P_{of}^{-1} guarantees a good performance in the high frequency range.

6 Example: low pass weights and delay systems

In this section we will consider a first order low pass weight and a plant with delay. We take the outer part of the plant to be strictly proper, so that the transfer function $\hat{P}(s)$ becomes continuous on the imaginary axis.

Let us choose the weight $\hat{W}$ and plant $\hat{P}$ to be

$$\hat{W}(s) = \frac{\varepsilon_w \tau_w s + 1}{\tau_w s + 1} \text{ and} \hat{P}(s) = \frac{1}{\tau_p s + 1}\frac{s-b}{s+b}e^{-hs}.$$

Here h is the amount of the time delay, $1/\tau_p$ is the bandwidth of the plant, and $1/\tau_w$ is the bandwidth of the weight (determining the band on which disturbance signals act). Typically ε_w is much less then 1. The conformal map between unit circle and right half plane will be taken as

$$z = \frac{\tau_w s - 1}{\tau_w s + 1} \quad \text{and} \quad s = \frac{1}{\tau_w}\frac{1+z}{1-z}.$$

This puts the weight in the form $W(z) = p(z) = p_0 + p_1 z$, with $p_0 = (1+\varepsilon_w)/2$ and $p_1 = -(1-\varepsilon_w)/2$. In this case we have $G(z) = \rho z/p(z)$ (see Appendix A).

A natural choice for $\hat{P}(s)$ is $\hat{m}_f(s)/(\tau_p s + 1)$, where $\hat{m}_f$ is a finite dimensional approximation of the inner part of the plant. For the inverse of the outer part we choose the proper function

$$\hat{P}_{of}^{-1}(s) = \frac{\tau_p s + 1}{\varepsilon_p \tau_p s + 1}$$

where $\varepsilon_p > 0$ is very small (we discuss later how small this should be). Now we check under which conditions the finite dimensional controller

$$C_f(z) = \left(\frac{p(z)\tilde{p}(z)}{\rho^2 z} - 1\right)\frac{G(z)P_{of}^{-1}(z)}{1 + G(z)P_f(z)P_{of}^{-1}(z)}$$

is approximately optimal. Recall the equation (6), from which we have

$$B_f = B_{opt}\frac{1}{1+X_1} + \frac{X_2}{1+X_1}$$

where $X_1 := \Delta/R$, and $X_2 = W\delta/R$. Therefore, if the conditions

$$(a):\ X_1 \in H^\infty;\ \ (b):\ X_2 \in H^\infty;\ \ (c):\ \|X_1\|_\infty < 1$$

are satisfied then $B_f \in H^\infty$. Assuming $m_f \in H^\infty$, then for (a) and (b) to hold it is necessary and sufficient to have

$$\hat{m}_f(j\omega_c) = \hat{m}(j\omega_c)(1 + j\varepsilon_p\tau_p\omega_c) \tag{7}$$

(see Appendix B), where ω_c is determined by the zeros of $(\tilde{p}(z)p(z) - \rho^2 z)$ for $z = (\tau_w s - 1)/(\tau_w s + 1)$. Simple computations give that

$$\omega_c = \frac{y}{\tau_w} \quad \text{with} \quad y = \sqrt{\frac{1-\rho^2}{\rho^2 - \varepsilon_w^2}}.$$

We will choose a Padé approximation for the delay term and add a filter to this to take into account the effect of $(1 + j\varepsilon_p\tau_p\omega_c)$:

$$\hat{m}_f(s) = \frac{s-b}{s+b}\hat{m}_d(s)\hat{F}(s)$$

where

$$\hat{F}(s) = \frac{s^2 + 2\omega_c s + (\omega_c^2 - r_c^2)}{s^2 + 2\omega_c s + \omega_c^2}$$

and $\hat{m}_d$ is a Padé approximation which is going to be defined below. The choice of $r_c = \omega_c\sqrt{2\varepsilon_p\tau_p\omega_c}$ makes $\hat{F}(j\omega_c) = (1 + j\varepsilon_p\tau_p\omega_c)$. So, we need only to check if, say the **first order**, Padé approximation $(1 - jh\omega_c/2)/(1 + jh\omega_c/2)$ is actually equal to $e^{jh\omega_c}$. This does not in general hold, however when h is order of magnitude .01 (and ω_c is less then .1) then the difference is so small (less then 10^{-10}) that we can fix the problem by changing the term $(1 - hs/2)/(1 + hs/2)$ to

$$K\,\frac{1 - (\frac{h}{2} + d_c)s}{1 + \frac{h}{2}s} =: \hat{m}_d(s).$$

Here for such small values of h and ω_c we have $1 - K$ and d_c are less then 10^{-10}. So, in the frequency range of interest (we will see that for such small h this is $0 \le \omega \le 10^2$) $\hat{m}_d(j\omega)$ can practically be seen to be equal to the first order Padé approximation of $e^{j\omega}$. In summary, we are going to use

$$\hat{m}_f(s) = \frac{s-b}{s+b}\hat{m}_d(s)\hat{F}(s)$$

in the controller, where $\hat{m}_d$ is a first order approximation for the delay term. This takes care of conditions (a) and (b).

One other condition we need to satisfy is $\|X_1\|_\infty < 1$. After substitution of the terms we see that

$$\hat{X}_1(s) = \frac{\frac{1}{\rho}\ \frac{\varepsilon_w\tau_w s - 1}{\tau_w s + 1}\ \frac{-\varepsilon_p\tau_p s}{\varepsilon_p\tau_p s + 1}\ \hat{m}(s)\ +\ \rho\ \frac{\tau_w s - 1}{\varepsilon_w\tau_w s + 1}\ \frac{\hat{m}_f(s) - \hat{m}(s)}{\varepsilon_p\tau_p s + 1}}{1 + \frac{1}{\rho}\ \frac{\varepsilon_w\tau_w s - 1}{\tau_w s + 1}\ \hat{m}(s)}$$

Define

$$\hat{R}_e(s) := 1 + \frac{1}{\rho}\ \frac{\varepsilon_w\tau_w s - 1}{\tau_w s + 1}\ \hat{m}(s),$$

and

$$\hat{D}_e(s) := \hat{X}_1(s)\hat{R}_e(s).$$

Plot $|\hat{R}_e(j\omega)|$ versus ω, and choose $\varepsilon_p\tau_p$ small enough such that

$$|\hat{D}_e(j\omega)| \ll |\hat{R}_e(j\omega)| \quad \text{for} \quad \omega \le \omega_r$$

ω_r is to be defined below. Consider the second term in the numerator of $\hat{X}_1$. Note that

$$\rho\,\left|\frac{\tau_w j\omega - 1}{\varepsilon_w\tau_w j\omega + 1}\right| \to \frac{\rho}{\varepsilon_w} \quad \text{as} \quad \omega \to \infty.$$

So, $\hat{m}_f$ should approximate $\hat{m}$ "reasonably good" at least up to frequency ω_r: where

$$\varepsilon_p\tau_p\omega_r \gg \frac{2\rho}{\varepsilon_w}.$$

Then, $|\hat{D}_e(j\omega)| \approx \varepsilon_w/\rho$ (which is going to be strictly smaller then $1 - \varepsilon_w/\rho \approx |\hat{R}_e(j\omega)|$ when $\rho > 2\varepsilon_w$) for $\omega \ge \omega_r$. We therefore assume that $\rho > 2\varepsilon_w$. All these guarantee only the stability of B_f.

At this point we should also check the performance, i.e. consider $|\hat{B}_f(j\omega)|$. Recall the expression

$$\hat{B}_f = \hat{B}_{opt}\frac{1}{1+\hat{X}_1} + \frac{\hat{X}_2}{1+\hat{X}_1}$$

where $\hat{X}_1$ is as above and $\hat{X}_2$ can be computed as

$$\hat{X}_2(s) = \rho\frac{\tau_w s - 1}{\tau_w s + 1}\ \frac{\hat{m}(s)\ \frac{-\varepsilon_p\tau_p s}{\varepsilon_p\tau_p s + 1}\ +\ \frac{\hat{m}_f - \hat{m}}{\varepsilon_p\tau_p s + 1}}{1 + \frac{1}{\rho}\ \frac{\varepsilon_w\tau_w s - 1}{\tau_w s + 1}\ \hat{m}(s)}.$$

Now, since at low frequencies $|\hat{X}_1(j\omega)| \ll 1$ and $|\hat{X}_2(j\omega)| \ll 1$, by sufficiently small ε_p and good approximation of m by m_f, we have

$$|\hat{B}_f(j\omega)| \approx |\hat{B}_{opt}(j\omega)|$$

for low frequencies. It is also not difficult to see that at high frequencies $|\hat{B}_f(j\omega)| \to \varepsilon_w < \rho$ because $|\frac{1}{j\varepsilon_p\tau_p\omega + 1}| \to 0$.

Thus recapping, choosing ε_p sufficiently small (for good performance at low frequencies) and approximating $\hat{m}(j\omega)$ by $\hat{m}_f(j\omega)$ up to the frequency range near $\omega_r \gg \frac{2\rho}{\varepsilon_w\varepsilon_p\tau_p}$, also satisfying the interpolation condition posed by stability, we have approximate optimality.

We remark about a trade-off now. That is, for good performance we need to use small ε_p; however, this increases the frequency band on which $\hat{m}$ should be approximated well, which forces us to use higher order approximations if the delay h is not small enough.

Let us look at a specific design example by choosing $\tau_w = 200$, $\tau_p = 100$, $\varepsilon_w = 0.05$, $h = 0.01$, $\varepsilon_p = 0.01$, and let b be such that $\rho = 0.2$. These make $\omega_c = 1/39.5$ and b is given by

$$b = \omega_c\left(\tan\left(\frac{\pi - h\omega_c - \tan^{-1}\varepsilon_w y\ - \tan^{-1}y}{2}\right)\right)^{-1} = 0.0267 \tag{8}$$

(see Appendix A). The values for d_c and K can be computed by equating the magnitude and phase of $e^{j\omega_c}$ to the magnitude and phase of $\hat{m}_d(j\omega_c)$. We find that $K = (1 + 5.530 \times 10^{-13})^{-1}$ and $d_c = 2.765 \times 10^{-11}$. The magnitude plots of $\hat{R}_e$, $\hat{D}_e$ and $\hat{X}_1$ are given in Figures 4, 5, 6 respectively; we observe the stability from these plots (note also that there is no unstable pole-zero cancellation in the controller and the plant). For the performance bound see Figure 7: the magnitude plot of $\hat{B}_f$. We have $\|\hat{B}_f\|_\infty = 0.208$, i.e. the deviation from the optimal performance is about 4 percent.

If better performance is desired then one should decrease the value of ε_p and refine the approximation of the delay term accordingly (if necessary).

Remark For plants with invertible outer parts we have seen in Section 4 that there is a difficulty in our method. Nevertheless, we can overcome this difficulty as follows. When the plant has an invertible outer part, for the design of H^∞ controllers, we can assume without loss of generality that $\hat{P} = \hat{m}$. Suppose that there exists a strictly proper rational outer transfer function $\hat{P}_e$ such that $\hat{P}_e\hat{m}$ is continuous on the imaginary axis. Since $\hat{P}_e$ is outer the optimal performance $\mu(\hat{m})$ corresponding to the plant $\hat{m}$ is the same as the optimal performance $\mu(\hat{P}_e\hat{m})$ for the plant $\hat{P}_e\hat{m}$. Moreover, the corresponding optimal H^∞ controllers are related as follows:

$$C_{opt}(\hat{m}) = C_{opt}(\hat{P}_e\hat{m})\hat{P}_e.$$

Let us try to find a finite dimensional H^∞ controller for the plant $\hat{m}$ by approximating $C_{opt}(\hat{P}_e\hat{m})$:

$$C_f(\hat{m}) = C_f(\hat{P}_e\hat{m})\hat{P}_e$$

where the structure of $C_f(\hat{P}_e\hat{m})$ is as in Figure 3, with P_e taking place of P_o. Also, note that the sensitivity function, when $C_f(\hat{m})$ is used for the plant $\hat{m}$, is exactly the same as the sensitivity function when $C_f(\hat{P}_e\hat{m})$ is used as a controller for the plant $\hat{P}_e\hat{m}$. In Sections 5 and 6 of this paper we have solved the problem corresponding to the situation where the plant is continuous, e.g. $\hat{P}_e\hat{m}$, and the controller is in the form of $C_f(\hat{P}_e\hat{m})$. So, in summary, for plants with invertible outer parts an approximately optimal finite dimensional controller can be found by introducing a rational outer function $\hat{P}_e$, which takes the role of the outer part and makes $\hat{P}_e\hat{m}$ continuous on the imaginary axis, and then finding a finite dimensional controller (for $\hat{P}_e\hat{m}$) using the theory of Sections 5 and 6. □

7 Conclusions

We have obtained the structure of all suboptimal H^∞ controllers for systems with rational weights and stable arbitrary distributed plants. From this structure we have identified the infinite and finite dimensional parts of these H^∞ controllers. Our main objective in this paper was to illustrate that an approximately optimal finite dimensional controller can be designed by appropriately approximating the infinite dimensional parts of the optimal controller.

An important open problem arising from our results is: Given $\rho > \mu$ characterize the set of all $u \in H^\infty$, $\|u\|_\infty \leq 1$, such that the transfer function

$$\frac{G_u}{1 + mG_u}$$

in (5a) is finite dimensional, and find the lowest possible dimension. Solution to this problem would give the characterization of **all finite dimensional** suboptimal H^∞ controllers for stable distributed plants.

8 Appendix A

Optimal Sensitivity for $W(z) := p(z) = p_0 + p_1 z$

We have seen that computation of the optimal sensitivity reduces to finding a nonzero vector h_o such that $A_\rho h_o = 0$. That is, in our case,

$$\left((p_0 + p_1 T)(p_0 + p_1 T^*) - \rho^2 I\right) h_o = 0. \qquad (a1)$$

But we also have the following

$$Th_o = zh_o(z) - m(z)u_{-1}, \qquad (a2.a)$$

$$T^*h_o = z^{-1}(h_o(z) - h_o(0)) \qquad (a2.b)$$

and

$$TT^*h_o = h_o(z) - h_o(0)(1 - m(z)m(0)) \qquad (a2.c)$$

for some constants u_{-1} and $u_1 := h_o(0)$. Putting these expressions in $(a1)$ we see that $(a1)$ is equivalent to

$$\left(z^2 + (\frac{p_0^2 + p_1^2 - \rho^2}{p_1 p_0})z + 1\right) h_o(z) = \frac{p(z)u_1}{p_0} + zm(z)(u_{-1} - \frac{m(0)u_1}{p_0}).$$

Recall that by Sarason's theorem $B_{opt} = \rho^2 q(T)^* h_o / p(T)^* h_o$. In our case we then have

$$B_{opt}(z) = \rho^2 \frac{zh_o(z)}{(zp_0 + p_1)h_o(z) - p_1 u_1}.$$

Set $\tilde{p}(z) = zp_0 + p_1$, $\lambda(z) = -(z^2 + (p_0^2 + p_1^2 - \rho^2)z/p_0 p_1 + 1)$, and $\hat{u}_{-1} := (u_{-1} - \frac{1}{p_0}m(0)u_1)$. Since

$$h_o(z) = \frac{p(z)u_1/p_0 + zm(z)\hat{u}_{-1}}{-\lambda(z)}, \qquad (a3)$$

in $B_{opt}(z)$, and arranging terms we get

$$B_{opt}(z) = p(z)\frac{1 + m(z)zp_0\hat{u}_{-1}/p(z)u_1}{1 + m(z)\tilde{p}(z)p_0\hat{u}_{-1}/\rho^2 u_1}. \qquad (a4)$$

On the other hand from equation $(a3)$ we obtain that

$$\frac{1}{p_0}p(\alpha_i)u_1 + \alpha_i m(\alpha_i)\hat{u}_{-1} = 0$$

for $i = 1, 2$ where α_1 and $\alpha_2 = \alpha_1^{-1}$ are the roots of $\lambda(z) = 0$ on the unit circle. So, we must have

$$p(\alpha_1)\alpha_2 m(\alpha_2) = p(\alpha_2)\alpha_1 m(\alpha_1) \qquad (a5)$$

in order to have a nonzero vector h_o satisfying $A_\rho h_o = 0$. Solving the equations, after tedious computations, one gets that $p_0\hat{u}_{-1}/u_1 = \rho$. Putting this result in $(a4)$, and solving for Q_{opt} from $B_{opt} = W - mQ_{opt}$; and then solving for the optimum controller C_{opt}, via Youla parametrization, we end up with

$$C_{opt} = \left(\frac{W(z)W(z^{-1})}{\rho^2} - 1\right)\frac{G(z)}{1 + G(z)m(z)}P_o^{-1}(z)$$

where $G(z) = \rho z/p(z)$. The computations are straightforward but too lengthy to present here.

When $\hat{m}(s) = \frac{s-b}{s+b}e^{-hs}$ we find ρ from the equation:

$$h\omega_c + \tan^{-1}\varepsilon_w y + \tan^{-1} y + 2\tan^{-1}(\omega_c/b) = \pi$$

where

$$\omega_c = \frac{y}{\tau_w} \quad \text{and} \quad y = \sqrt{\frac{1 - \rho^2}{\rho^2 - \varepsilon_w^2}}.$$

This is obtained by writing the equation $(a5)$ explicitly and transforming the data from the unit circle to right half plane using the transformation $z = \frac{\tau_w s - 1}{\tau_w s + 1}$. Hence in the example considered when τ_w, ε_w, ρ and h are fixed b is found by (8).

9 Appendix B

Recall the expression

$$\hat{X}_1(s) = \frac{\frac{1}{\rho}\ \frac{\varepsilon_w\tau_w s - 1}{\tau_w s + 1}\ \frac{-\varepsilon_p\tau_p s}{\varepsilon_p\tau_p s + 1}\ \hat{m}(s) \ + \ \rho\ \frac{\tau_w s - 1}{\varepsilon_w\tau_w s + 1}\ \frac{\hat{m}_f(s) - \hat{m}(s)}{\varepsilon_p\tau_p s + 1}}{1 + \frac{1}{\rho}\ \frac{\varepsilon_w\tau_w s - 1}{\tau_w s + 1}\ \hat{m}(s)}$$

and definitions

$$\hat{R}_e(s) := 1 + \frac{1}{\rho}\ \frac{\varepsilon_w\tau_w s - 1}{\tau_w s + 1}\ \hat{m}(s),$$

$$\hat{D}_e(s) := \hat{X}(s)\hat{R}_e(s).$$

Also from the equations $(a3.a)$, $(a4)$ and $(a5)$ of Appendix A it is easy to see that

$$\frac{1}{\rho}\ \frac{\varepsilon_w\tau_w j\omega_c - 1}{\tau_w j\omega_c + 1}\ \hat{m}(j\omega_c) = -1 \qquad (b1)$$

Moreover, $+j\omega_c$ and $-j\omega_c$ are the only points where $\hat{R}_e(s)$ vanishes in the closed right half plane. Therefore for the stability of $\hat{X}_1$ we need

$$\hat{D}_e(j\omega_c) = 0. \qquad (b2)$$

Using $(b1)$ and re-arranging terms we get that $(b2)$ is equivalent to having

$$(-1)(-\varepsilon_p\tau_p j\omega_c) + (-\frac{1}{\hat{m}(j\omega_c)})(\hat{m}_f(j\omega_c) - \hat{m}(j\omega_c)) = 0,$$

or

$$\hat{m}_f(j\omega_c) = \hat{m}(j\omega_c)(1 + \varepsilon_p\tau_p j\omega_c). \qquad (b3)$$

It is also routine to check that

$$\frac{\partial}{\partial s}(\frac{\varepsilon_w\tau_w s - 1}{\tau_w s + 1})\ \hat{m}(s)|_{s=j\omega_c} \neq 0. \qquad (b4)$$

So, since $+j\omega_c$ and $-j\omega_c$ are the only points in the closed right half plane that makes $\hat{R}_e(s) = 0$, condition $(b3)$ is also sufficient for the stability of $\hat{X}_1$.

Now let's look at $\hat{X}_2$:

$$\hat{X}_2 = \rho\frac{\tau_w s - 1}{\tau_w s + 1}\frac{1}{\varepsilon_p\tau_p s + 1}\frac{\hat{m}(s)(-\varepsilon_p\tau_p s) + (\hat{m}_f(s) - \hat{m}(s))}{1 + \frac{1}{\rho}\ \frac{\varepsilon_p\tau_p s - 1}{\tau_w s + 1}\ \hat{m}(s)}.$$

From this expression, and the arguments used for $\hat{X}_1$ we see that the stability of $\hat{X}_2$ also is equivalent to $(b3)$.

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FIGURES

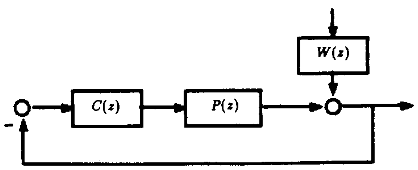

Figure 1: Closed Loop Control System

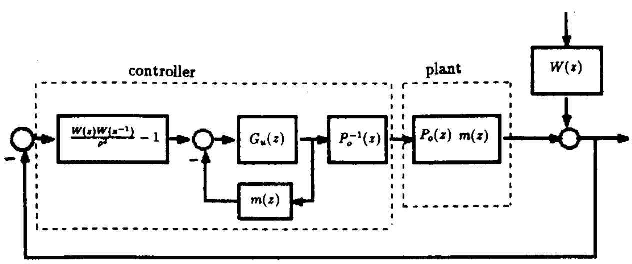

Figure 2: Structure of the Suboptimal Controller

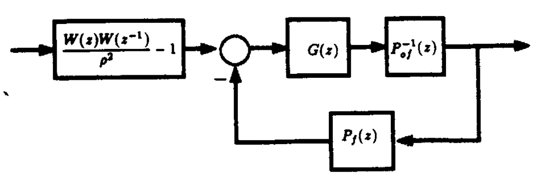

Figure 3: Finite dimensional controller

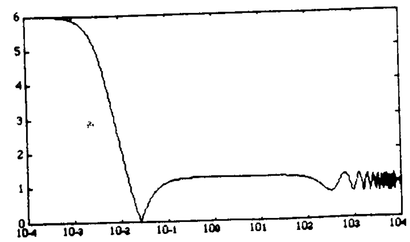

Figure 4: $|\hat{R}_e(j\omega)|$

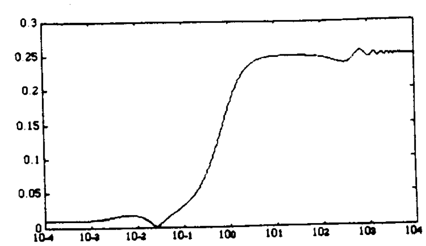

Figure 5: $|\hat{D}_e(j\omega)|$

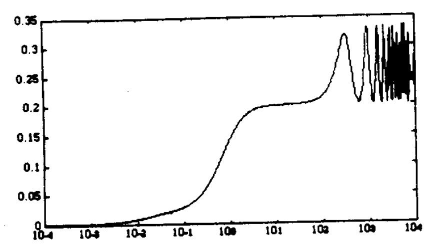

Figure 6: $|\hat{X}_1(j\omega)|$

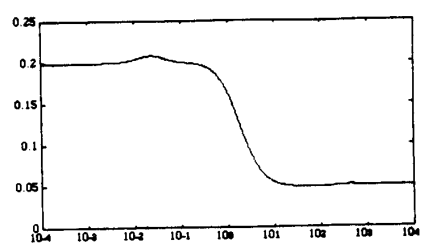

Figure 7: $|\hat{B}_f(j\omega)|$